Lecture 6
Modelling Second Order Systems and Examples

Process Control
Prof. Kannan M. Moudgalya

IIT Bombay
Wednesday, 31 July 2013

Outline

1. **Poles, zeros, response for different pole locations**
2. **Solution to second order systems**
3. **Integrating/capacitive systems**

1. Poles, zeros, response for different pole locations

Transfer function terminologies

- **Let the transfer function be $G(s) = N(s)/D(s)$ where $N(s)$ and $D(s)$ are polynomials in s**
- **The roots of $N(s) = 0$ are called *zeros* of $G(s)$**
- **The roots of $D(s) = 0$ are called *poles* of $G(s)$**

Transfer Function

The a_i in the following transfer function are known as

$$G(s) = \frac{(s - a_1)\cdots(s - a_m)}{(s - b_1)\cdots(s - b_n)}$$

1. **Poles**
2. **Zeros**
3. **Gain**
4. **Time constants**

Answer: 2. Zeros

Transfer Function

The b_j in the following transfer function are known as

$$G(s) = \frac{(s - a_1) \cdots (s - a_m)}{(s - b_1) \cdots (s - b_n)}$$

1. Poles
2. Zeros
3. Gain
4. Time constants

Answer: 1. Poles

Poles and zeros of two transfer functions

- **Let the transfer function be $G(s) = N(s)/D(s)$ where $N(s)$ and $D(s)$ are polynomials in s**
- **The poles of $1 + KG(s)$ are**
  1. **same as the poles of $G(s)$**
  2. **same as the zeros of $G(s)$**
  3. **No relation between the poles/zeros of $G(s)$ and $1 + KG(s)$**

Answer: 1. same as the poles of $G(s)$

Poles and zeros of two transfer functions

- Let the transfer function be $G(s) = N(s)/D(s)$ where $N(s)$ and $D(s)$ are polynomials in s
- Let the closed loop transfer function be $T(s)$:

$$T(s) = \frac{KG(s)}{1 + KG(s)}$$

- The poles of $T(s)$ are
  1. Poles of $G(s)$
  2. Zeros of $G(s)$
  3. Zeros of $D(s) + KN(s)$
  4. Poles of $D(s) + KN(s)$

Answer: 3. Zeros of $D(s) + KN(s)$

Partial fraction of second order system

- $Y(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{M}{s} = \frac{A}{s+a} + \frac{B}{s+b} + \frac{C}{s}$

Given that y(t) is the output of a real life plant, and a is complex,

1. **There is no relationship between a and b**
2. **b is real**
3. **$a = -b$**
4. **a and b are complex conjugates**

Answer: 4, i.e. a and b are complex conjugates
If a and b are not complex conjugates,
y(t) will be imaginary, not realistic for a real plant!

Partial fraction of second order system

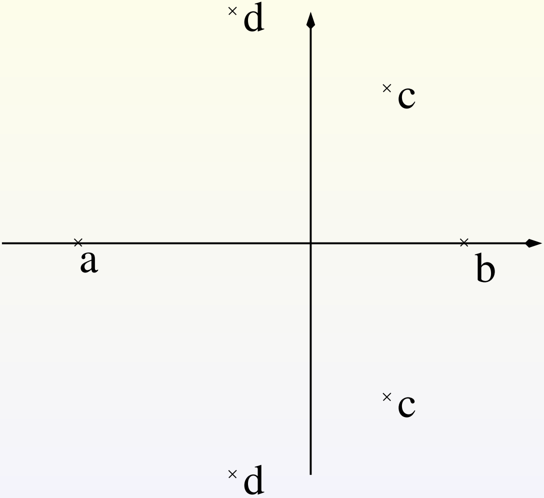
- $$Y(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}\frac{M}{s} = \frac{A}{s+a} + \frac{B}{s+b} + \frac{C}{s}$$

Given that y(t) is the output of a real life plant, and A is complex,

1. **There is no relationship between A and B**
2. **B is real**
3. **$A = -B$**
4. **A and B are complex conjugates**

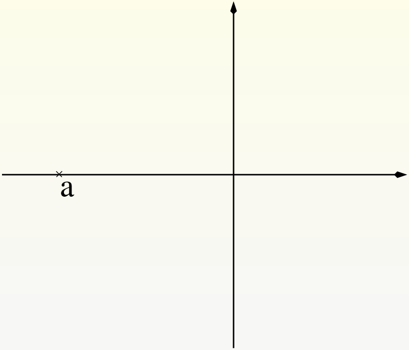
Answer: 4, i.e. A and B are complex conjugates
If A and B are not complex conjugates,
y(t) will be imaginary, not realistic for a real plant!

Response for different pole locations



Poles are indicated by crosses

Step response for pole at a

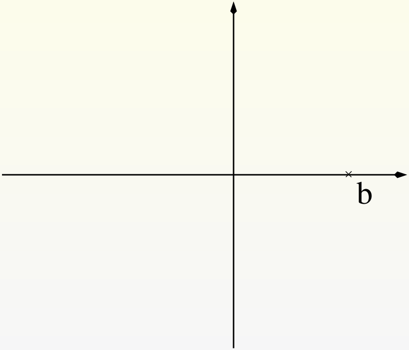


- **For the pole at a place indicated by a, the response is of the form $e^{-\alpha^2 t}$**
- **The exponential part will decay, reaching a constant value**

Scilab code to plot step response of a negative pole, a.sce

```
s = %s
G_a = 1/(s+3)
t = 1:0.1:5;
y_a = csim('step',t,G_a);
plot2d(t,y_a)
```

Response for pole at b

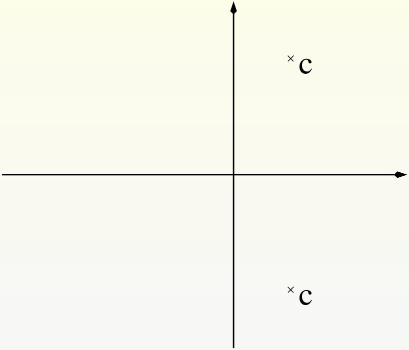


- **For the pole at a place indicated by b, the response is of the form $e^{\beta^2 t}$**
- **The exponential part will grow unbounded**

Scilab code to plot step response of a positive pole, b.sce

```
s = %s
G_b = 1/(s-2)
t = 0:0.01:6;
y_b = csim('step',t,G_b);
plot2d(t,y_b)
```

Response for pole at c

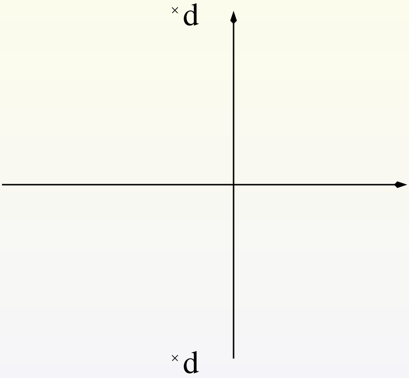


- For the poles at places indicated by c, the response is of the form $e^{\gamma^2 t} \times$ (sinusoidal terms)
- There will be growing oscillations

Scilab code to plot step response of complex conjugate poles in right half plane, c.sce

```
s = %s
G_c = 1/(s-(1+2*%i))/(s-(1-2*%i))
t = 0:0.01:10.15;
y_c = csim('step',t,G_c);
plot2d(t,y_c)
```

Response for poles at d



- For the poles at places indicated by d, the response is of the form $e^{-\delta^2 t} \times$(sinusoidal terms)
- There will be decaying oscillations

Scilab code to plot step response of complex conjugate poles in left half plane, d.sce

```
s = %s
G_d = 1/(s-(-1+3*%i))/(s-(-1-3*%i))
t = 0:0.01:6;
y_d = csim('step',t,G_d);
plot2d(t,y_d)
```

2. Second order system

Second order system

- $Y(s) = G(s)U(s)$, $G(s) = \dfrac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$
- τ is the time constant
- ζ is the damping coefficient
- K is the steady state gain
- Step response: $Y(s) = \dfrac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}\dfrac{M}{s}$
- $= \dfrac{A}{s+a} + \dfrac{B}{s+b} + \dfrac{C}{s}$
- Is there a relation between a and b, A and B?
- For $\zeta < 1$, $a = b^*$, $A = B^*$

Step response of a second order system

- $Y(s) = \dfrac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}\dfrac{M}{s} = \dfrac{A}{s+a} + \dfrac{B}{s+b} + \dfrac{C}{s}$
- a and b are roots of $\tau^2 s^2 + 2\zeta\tau s + 1 = 0$
- $a, b = \dfrac{-2\zeta\tau \pm \sqrt{4\zeta^2\tau^2 - 4\tau^2}}{2\tau^2}$
- $= -\dfrac{\zeta}{\tau} \pm \dfrac{1}{\tau}\sqrt{\zeta^2 - 1}$
- $\zeta \geq 1$, a, b are real
- $\zeta < 1$, complex, $a = b^*$, $A = B^*$

Step response for $\zeta > 1$

$$Y(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}\frac{M}{s}$$

- Known as an overdamped system ($\zeta > 1$)
- Its solution is given by

$$y(t) = KM\left\{1 - e^{(-\zeta t/\tau)} *\right.$$
$$\left.\left[\cosh\left(\frac{\sqrt{\zeta^2-1}}{\tau}t\right) + \frac{\zeta}{\sqrt{\zeta^2-1}}\sinh\left(\frac{\sqrt{\zeta^2-1}}{\tau}t\right)\right]\right\}$$

Step response for $\zeta = 1$

$$Y(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}\frac{M}{s}$$

- Known as critically damped system ($\zeta = 1$)
- Its solution is given by

$$y(t) = KM\left[1 - \left(1 + \frac{t}{\tau}\right)e^{-t/\tau}\right]$$

Step response for $\zeta < 1$

$$Y(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}\frac{M}{s}$$

- **Known as underdamped system ($\zeta < 1$)**
- **Its solution is given by**

$$y(t) = KM\left\{1 - e^{(-\zeta t/\tau)} *\right.$$

$$\left.\left[\cos\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right) + \frac{\zeta}{\sqrt{1-\zeta^2}}\sin\left(\frac{\sqrt{1-\zeta^2}}{\tau}t\right)\right]\right\}$$

Another form of II order system

$$G(s) = \frac{K}{\tau^2 s^2 + 2\zeta\tau s + 1}$$

will also be written as

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

- **K is taken as 1**
- ω_n**: natural frequency,** ζ**: damping factor.**
- **For different ζ, get underdamped ($\zeta < 1$), critically damped ($\zeta = 1$) and overdamped ($\zeta > 1$) systems**

You should be able to derive all the previous expressions for $\mathbf{y(t)}$

Solution to underdamped system

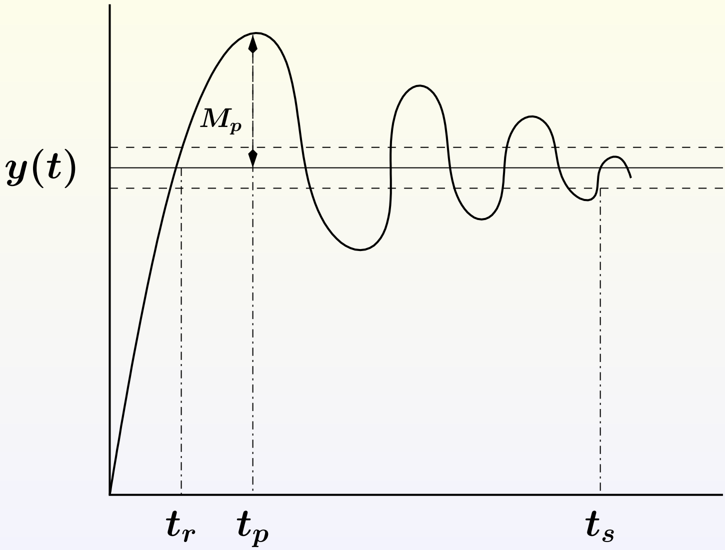
Underdamped second order system:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

For $\zeta < 1$, rts of den. $= -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2}$
The step response is,

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left[\omega_n\sqrt{1-\zeta^2}\,t + \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}\right]$$

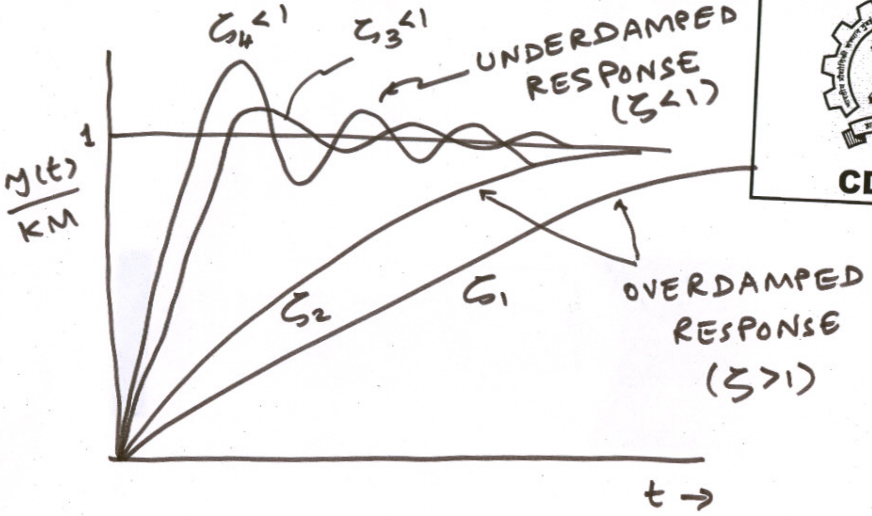
Step response of underdamped system



Scilab code step-II-order.sce

```
exec('second.sci');
s = %s
omegan = 1; t = 1:0.1:15;
y = second(omegan,0.1,t);
y = second(omegan,0.5,t);
y = second(omegan,1,t);
y = second(omegan,2,t);

function y = second(omegan,zeta,t)
G = omegan^2/(s^2+2*zeta*omegan*s+
   omegan^2)
y = csim('step',t,G);
plot2d(t,y)
endfunction
```

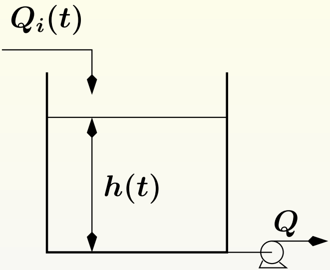


$\zeta_4 < \zeta_3 < \zeta_2 < \zeta_1$

$\zeta = 1 \Rightarrow$ CRITICALLY DAMPED

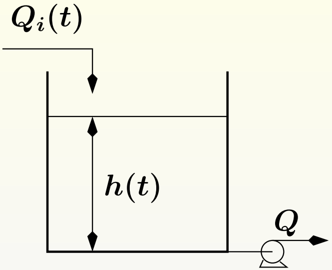
3. Integrating/capacitive systems

Integrating Process



- $A\dfrac{dh}{dt} = Q_i - Q$
- In deviational variables,
- $A\dfrac{d\Delta h}{dt} = \Delta Q_i(t) - \Delta Q(t)$
- Taking Laplace transform,
  $As\Delta h(s) = \Delta Q_i(s) - \Delta Q(s)$
- $\Delta h(s) = \dfrac{1}{As}\Delta Q_i(s) - \dfrac{1}{As}\Delta Q(s)$

Integrating Process



In case of a step disturbance in feed flow, tank will

1. **overflow**
2. **run dry**
3. **overflow or run dry**
4. **stabilise at some suitable value**

Answer: 3: overflow or run dry

Ramp input to a first order system

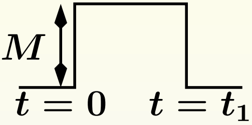
$$Y(s) = \frac{K}{\tau s + 1} U(s), \;\; U(s) = \frac{a}{s^2} \leftrightarrow at$$

- $Y(s) = \dfrac{K}{\tau s + 1}\dfrac{a}{s^2} = \dfrac{A}{\tau s + 1} + \dfrac{B}{s} + \dfrac{C}{s^2}$
- Multiply by $\tau s + 1$, let $s = -1/\tau$, $A = Ka\tau^2$
- Multiply by s^2:
- $\dfrac{Ka}{\tau s + 1} = \dfrac{A}{\tau s + 1} s^2 + Bs + C$
- By letting $s = 0$, $C = Ka$
- Differentiating blue equation w.r.t. s
- and letting $s = 0$, $B = -Ka\tau$

Final value of ramp input to a first order system

- $Y(s) = \dfrac{A}{\tau s + 1} + \dfrac{B}{s} + \dfrac{C}{s^2}$
- $Y(s) = \dfrac{Ka\tau^2}{\tau s + 1} - \dfrac{Ka\tau}{s} + \dfrac{Ka}{s^2}$
- Inverting
- $y(t) = Ka\tau\left(e^{-t/\tau} - 1\right) + Kat$
- $y(t = \infty) = Ka(t - \tau)$
- Let $K = 1$ and plot $u(t)$ and $y(t)$ in the same plot

Rectangular Pulse



- Give the following input, u(t):
- $u(t) = 1(t) \times M - 1(t - t_1) \times M$
- 1(t) denotes unit step input
- $u(t) = [1(t) - 1(t - t_1)] \times M$
- $U(s) = \left[\frac{1}{s} - \frac{1}{s}e^{-t_1 s}\right] M$
- $U(s) = [1 - e^{-t_1 s}]\, \frac{M}{s}$

What we learnt today

- **Poles, zeros, response of systems for different pole locations**
- **Solution to second order systems**
- **Integrating or capacitive systems**

Thank you