EP 222: Classical Mechanics - Lecture 5

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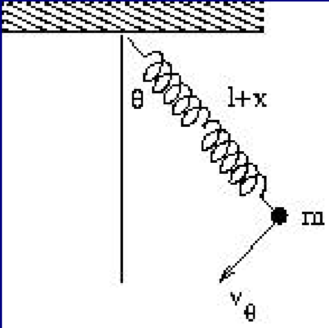
Review of Lecture 4: Euler Lagrange Equation

- We discussed application of the Euler Lagrange Equation to solve the Brachistochrone problem.
- Brachistochrone is the path for which the time for a bead to slide along a vertical track from a given point to another for which the time taken is the shortest.
- By minimising a quantity similar to action, we minimised the time taken and found the shape of the path is part of a cycloid.
- Today we will solve the spring pendulum problem using the E-L equations and then have a re-look at the E-L equations using the principle of virtual work.

Spring Pendulum

$$T = \frac{1}{2}m[\dot{x}^2 + (l+x)^2\dot{\theta}^2]$$

$$V = -mg(l+x)\cos\theta + \frac{1}{2}kx^2 \qquad \mathcal{L} = T - V$$



Spring Pendulum

- Take x and θ to be generalized coordinates.
- E-L equation for x

$$m\ddot{x} = m(l + x)\dot{\theta}^2 + mg\cos\theta - kx$$

First term centripetal acceleration term , second the component of mg along the spring, $-kx$ spring force. In a rotating frame of reference, the first term is the centrifugal force term.

Spring Pendulum

- E-L equation for θ

$$\frac{d}{dt}[m(l+x)^2\dot{\theta}] = -mg(l+x)\sin\theta$$

on the right is the torque about the point of suspension. L.h.s. gives the rate of change of angular momentum

$$L = m\vec{r}\times\vec{v} = m(l+x)\cdot(l+x)\dot{\theta} = m(l+x)^2\dot{\theta}$$

- Explicitly differentiating, we have

$$m(l+x)\ddot{\theta} + 2m\dot{x}\dot{\theta} = -mg\sin\theta$$

In a rotating frame it is the expression for the tangential acceleration term, the second term representing the Coriolis force.

Constraints

- Consider a system described by generalized coordinates $q_1, q_2, \ldots, q_m$, which may not all be independent because of constraints.
- Holonomic Constraints are those where algebraic relationship (s) exists between the coordinates. If there are k constraints, there are k equations of the type

$$f^i(t, 1, q_1, q_2, \ldots, q_m) = 0, \quad 1 \leq i \leq k$$

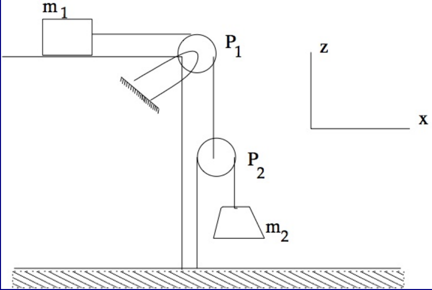
- Constraints are called Scleronomic if there is no explicit time dependence.
- Time dependent constraints are called Rheonomic.

Constraints

- Constraints which are not expressed as algebraic relations but are expressed as differential equations which constrain the generalized coordinates and velocities are called kinematic constraints.
- Non-integrable kinematic constraints cannot be reduced to holonomic constraints and are called Non-Holonomic.

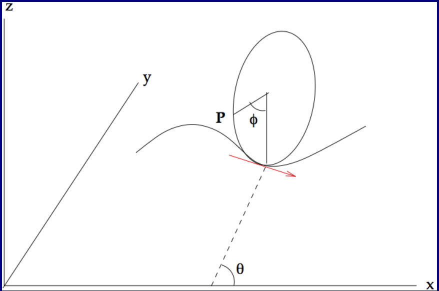
Holonomic Constraints

- Two masses connected by pulleys. Two particles $\implies$ 6 degrees of freedom.
- m_1 moves only along x and m_2 only along y direction, making the dof as 2, as $y_1 = z_1 = 0$ and $x_2 = y_2 = 0$.
- If m_1 moves to right by d, m_2 moves downwards by $2d$, one constraint. The problem is one dimensional.



Non-Holonomic Constraints

- Relationship expressed as algebraic inequality e.g. A mass moving inside a sphere $x^2 + y^2 + z^2 \leq 0$.
- Disc rolling without slipping on horizontal $x - y$ plane, keeping its own plane vertical.
- Four degrees of freedom: position (x, y) of the centre of mass, ϕ by which the disk has rotated about its axis of rotation and the angle θ that the axis makes with the x-axis.

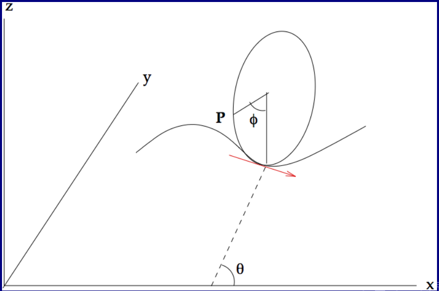


Non-Holonomic Constraints- Rolling Disc

- The axis makes θ with x-axis, so linear velocity v of the disk makes θ with y axis

$$\frac{dx}{dt} = v\sin\theta = R\omega\sin\theta = R\dot{\phi}\sin\theta$$
$$\frac{dy}{dt} = -v\cos\theta = -R\dot{\phi}\cos\theta$$

- Combining $dx = R\sin\theta d\phi$ and $dy = -r\cos\theta d\phi$. These equations cannot be reduced further.



Virtual Displacement

- A displacement of the system (probably imagined) consistent with the constraints on the system
- Displacements are instantaneous as no velocities are involved but the new configuration arrived at is a possible geometrical state of the system.
- If the system is in equilibrium, force $\vec{F}_i$ acting on each particle is zero.

Principle of Virtual Work

The work done by applied forces in a virtual displacement of a body is zero.

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- $\vec{F}_i = 0 \implies \sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0$

Principle of Virtual Work

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- $\vec{F}_i = 0 \implies \sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0$
- $F_i = F_i^a + F_i^c$ but the constraint forces (reaction, tension etc.) do not do any work.
- Thus $\sum_i \vec{F}_i^a \cdot \delta r_i = 0$