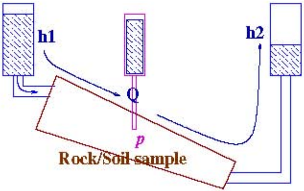
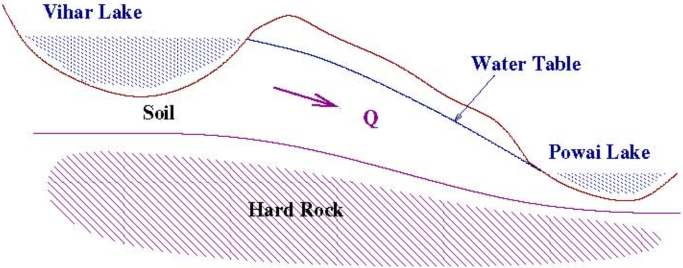
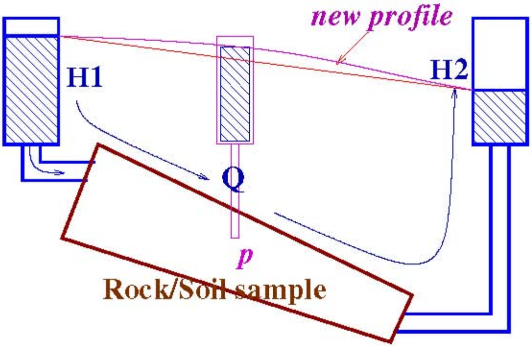
Not entirely fictitious



Varying soil thickness!

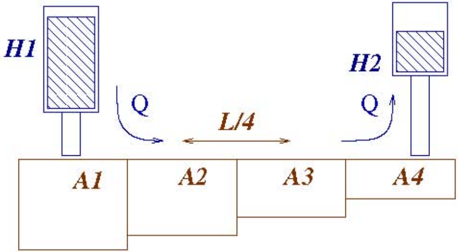


Coming back: A calculation



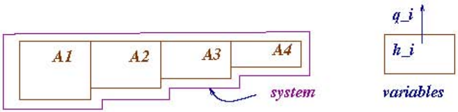
- Notice the change in thickness. Notice that if the thickness is large, the drop is the head is small.
- $\Delta H_1 * KA_1/L = Q = \Delta H_2 * KA_2/L$

Solving the narrowing pipe 1: Domain Decomposition



- Approximate the system in terms of simpler cells.
- For each cell, associate the internal variable, head h_i, and the external variable q_i, the external flow coming into the cell.
- Write Darcy's law and conservation law for each cell.

Posing the narrowing pipe



- Approximate the system in terms of simpler cells.
- For each cell, associate the internal variable, head h_i, and the external variable q_i, the external flow coming into the cell.
- Write Darcy's law (Note that last equation is superfluous).

$$
\begin{aligned}
q_1 = -q = -q_4, q_2 = q_3 &= 0 \\
h_1 = H_1, h_4 &= H_2 \\
(h_2 - H_1)KA_2/\ell &= -q \\
(H_1 - h_2)KA_2/\ell + (h_3 - h_2)KA_3/\ell &= 0 \\
(h_2 - h_3)KA_3/\ell + (H_4 - h_3)KA_4/\ell &= 0 \\
(h_3 - H_4)KA_4/\ell &= q
\end{aligned}
$$

Solving

Lets put $A_2 = 4, A_3 = 3, A_2 = 2$ and $K/\ell = \alpha$ to get:

$$\begin{aligned} (h_2 - H_1)4\alpha &= -q \\ (H_1 - h_2)4\alpha + (h_3 - h_2)3\alpha &= 0 \\ (h_2 - h_3)3\alpha + (H_4 - h_3)2\alpha &= 0 \end{aligned}$$

We get:

$$\begin{bmatrix} 7 & -3 \\ -3 & 5 \end{bmatrix} \begin{bmatrix} h_2 \\ h_3 \end{bmatrix} = \begin{bmatrix} 4H_1 \\ 2H_4 \end{bmatrix}$$

$$\frac{1}{26} \cdot \begin{bmatrix} 5 & 3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} 4H_1 \\ 2H_4 \end{bmatrix} = \begin{bmatrix} h_2 \\ h_3 \end{bmatrix}$$

Lets put $H_1 = 20$ and $H_4 = 10$, to get $h_2 = 17.7$ and $h_3 = 14.6$. Now solve for q.

Indicates a general method: Domain Decomposition. Break into n pieces and improve accuracy.

Analytically too...

- Let the cross-section be $A(x)$ at a distance$x$ from the left and let the head be $h(x)$. Whence we have:

$$A(x) = \frac{2 \cdot x + 5 \cdot (L - x)}{L}$$

- We also have:

$$Q = \frac{(h(x + \Delta x) - h(x)) \cdot A(x) K}{\Delta x}$$

- In other words, we have the following equation with the conditions that $h(0) = H_1$ and $h(L) = H_2$.

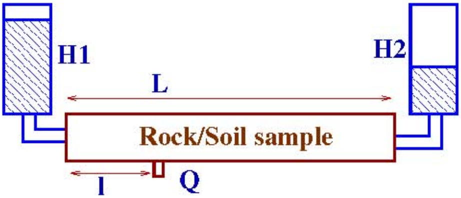
$$\frac{dh}{dx} = \frac{Q}{K \cdot A(x)}$$

- This may be solved and compared with the discrete version.
- However, most problems *cannot* be solved analytically.

Another problem

A horizontal pipe of length L and cross-section A is held at heads H_1 and H_2 at its ends. At a distance l from the left end, a tap is made to the pipe and a pump is fitted from which Q cu.m./s is extracted. The conductivity of the soil is K.

Find the head at the tapping point. What fraction of Q is coming from the left and what from the right? Can Q be increased without limits? Can you identify a real-life situation?



Analytic solution

- Let head at point p at distance ℓ be h. We have

$$\frac{(H_1 - h)AK}{\ell} - \frac{(h - H_2)KA}{L - \ell} = Q$$

- In other words

$$\frac{H_1}{\ell} + \frac{H_2}{L - \ell} - \frac{Q}{KA} = (\frac{1}{\ell} + \frac{1}{L - \ell}) \cdot h$$

- Finally, we have:

$$h = \frac{H_1(L - \ell) + H_2\ell}{L} - \frac{Q \cdot \ell \cdot (L - \ell)}{KAL}$$

- Many interesting things about this expression.

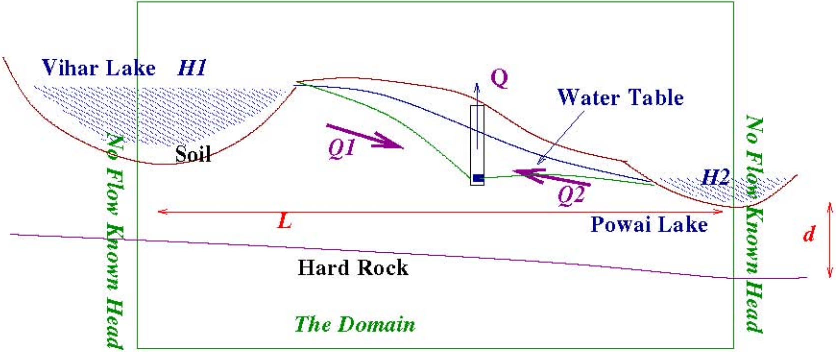
More details

What part comes from the H_1 and the H_2 side?

$$Q_1 = (H_1 - H_2)\frac{KA}{L} + Q\frac{L-\ell}{L}$$
$$Q_2 = (H_2 - H_1)\frac{KA}{L} + Q\frac{\ell}{L}$$

- We see that for $H_1 > H_2$, for small Q, $Q_2 < 0$, i.e., the tap merely diverts some of the flow going from H_1 to H_2.
- Only when Q is so large that h drops below H_2, that H_2 is giving some water to the tap.

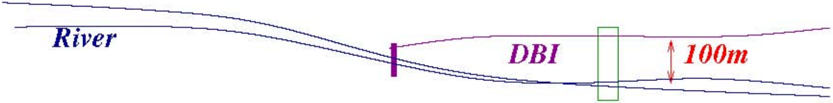
How does this help us in real life?



- If $L >> d >> H_1 - H_2$ then A is roughly constant and the drop in heads is small as compared to the depth of the aquifer.
- The deep aquifer assumption.
- The domain with known *boundary* conditions.
- Such systems are easily solvable.

DBI system

A DBI breaks off from a river and maintains an average distance of 100m from the river. Moreover, it is roughly 5m higher than the river. If at the entry point, the flow is 100*lps*, how much area can the DBI benefit?



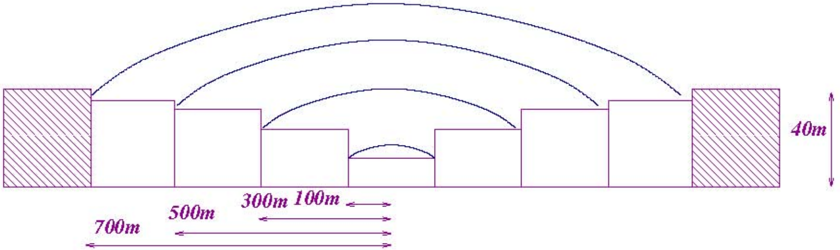
- Let us assume that aquifer thickness is $20m$ and $K = 1m/d$.
- Consider a domain of thickness 1m crossing the canal and the river.
- The flow from the canal to the river will be $Q = 5 \cdot 20/100 = 1$, i.e., 1 cubic meter per day flows from 1m of canal.
- The canal has 100lps=8600 cu.m./s.
- Thus by 8km, the flow in the canal will be reduced to zero.
- How much is the ET for this area?

The Approximate Well

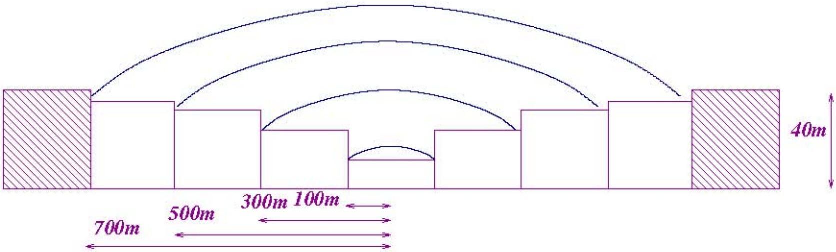
A well of depth $20m$ is being planned to irrigate a land of area 1 hectare. The ambient water table is $5m$ below ground water and the conductivity is $10m/d$. The aquifer depth is $45m$. Will the well provide the water? Let us assume that the bottom of the aquifer is a elevation 0.

We partition the domain into concentric rings, each of width $200m$.

We assume that the impact of the well is zero after $800m$.



The Approximate Well



- We have the variables $h(800), h(600), h(400), h(200)$ and $h(0)$, where the final variable is the head at the center, i.e., at the well.
- We also have that *far away*, $h(800) = 45 - 5 = 40$

Computing ...

- Lets assume $Q = 50 cu.m./day$.
- If we know $h(r+\Delta)$, then we may write:

$$\frac{K \cdot (2\pi r \times h(r)) \times (h(r+\Delta) - h(r))}{\Delta} = Q$$

- Takin $\pi = 25/8$, we have:

$$\frac{(500/8) \cdot h(r)) \times (h(r+200) - h(r))}{200} = 50$$

- This is a bit inconvenient, since it is quadratic in $h(r)$. Let us approximate $h(r)$ by $h(r+\Delta)$ and denote the difference by Δh to get:

$$h(r+200)) \times \Delta h = 160$$

Finally...

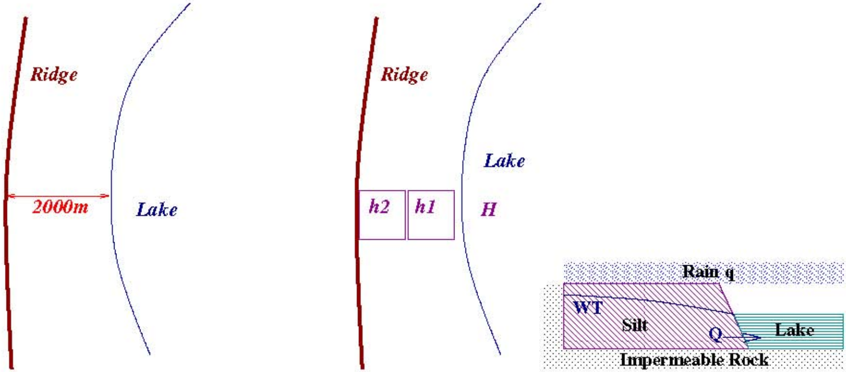
- Thus, we get $h(800) \cdot \Delta h = 160$, t oegt $\delta h = 4$ and thus $h(600) = 36$.
- We have $36 \times |deltah = 160$, to get $\Delta h = 4.5$ and $h(400) = 31.5$.
- Δh now becomes 5.1 and $h(200) = 26.4$.
- Finally, we have $\Delta h = 6.1$ with $h(0) = 20.3$.

Thus, the well must operate roughly $5 + (40 - 20.3) = 24.7m$ below ground level. The chosen depth (of $20m$) of the well is insufficient.

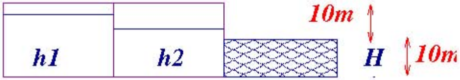
- Examine what happens when K is changed.
- What happens when the *more accurate* quadratic is solved.
- There is an analytic model as well!
- What happens if there is a neighboring well?

Lake and Ridge

A lake has a ridge on one side, $2km$ away. During the monsoon, it rains about $10mm$ per day with an infiltration of 2%. What is the height of the water table between the lake and the ridge, in the steady state. Assume suitable soil thickness.



A simple model



- Divide the domain into two squares , $1km \times 1km$, with heads h_1 and h_2. Thus the net infiltration in each cell 200 cu.m./day.
- *Assuming all flows are underground*, we have:

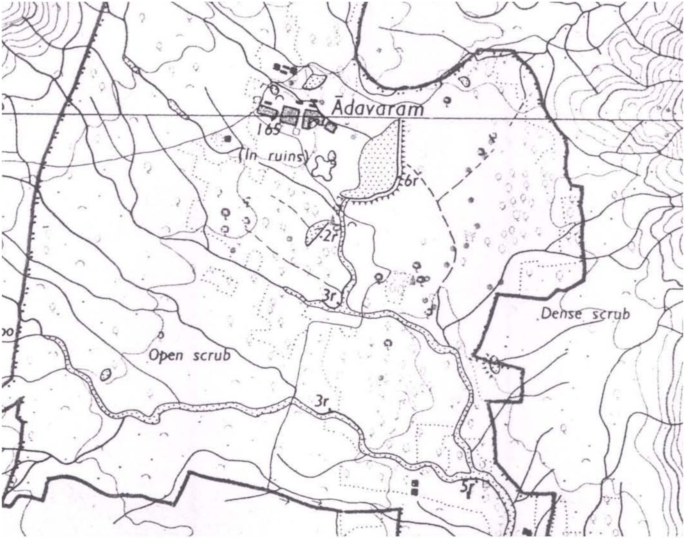
$$\frac{(h_2 - 10) \cdot K \cdot 1000 \cdot 10}{500} = 400$$

- This gives us $(h_2 - 10) \cdot K = 20$. If $K = 5$, we see $h_2 = 14$.
- The next equation gives $h_1 = 17$.

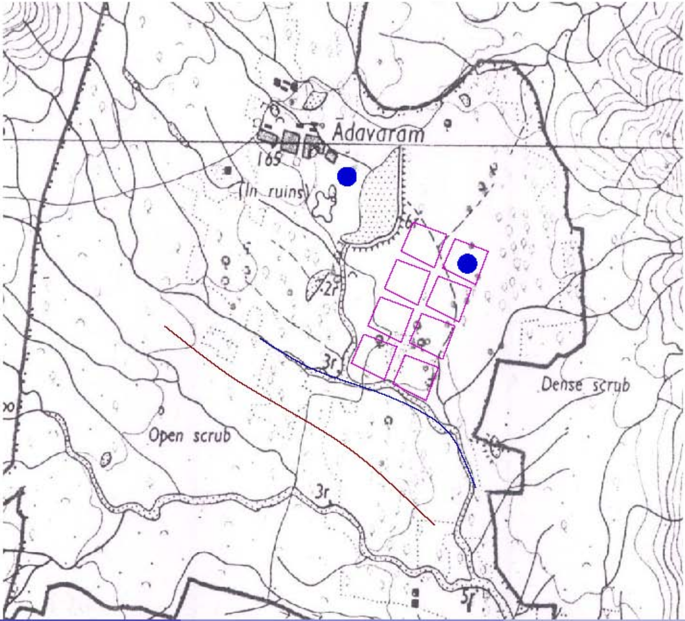
$$\frac{(h_1 - 14) \cdot 5 \cdot 1000 \cdot 14}{1000} = 200$$

- What would happen if the infiltration were greater? Or K were lower?

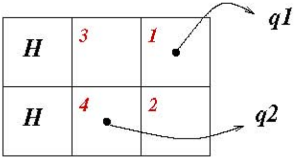
Real life...



Adjacent farms



The model



- Let us write the conservation equations.
- Denote $KA/L = \alpha$ and $Q_i = q_i/\alpha$.
- *Deep aquifer.* A does not depend on h.

- 6 squares, each side 1 km. The left-most adjoin the river, at a fixed head H.
- Two wells in two diagonal squares.
- What is the impact of the wells on each other?

$$
\begin{aligned}
(h_3 - h_1) + (h_2 - h_1) &= Q_1 \\
(h_1 - h_2) + (h_4 - h_2) &= 0 \\
(H - h_3) + (h_4 - h_3) + (h_1 - h_3) &= 0 \\
(H - h_4) + (h_3 - h_4) + (h_2 - h_4) &= Q_2
\end{aligned}
$$

$$
\begin{bmatrix} -2 & 1 & 1 & 0 \\ 1 & -2 & 0 & 1 \\ 1 & 0 & -3 & 1 \\ 0 & 1 & 1 & -3 \end{bmatrix}
\begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \end{bmatrix}
=
\begin{bmatrix} Q_1 \\ 0 \\ -H \\ Q_2 - H \end{bmatrix}
$$

Let us denote this equation as $Gh = b$.

Whence...

Thus $h = G^{-1}b$ where G is actually:

$$\frac{1}{11}\begin{bmatrix} -13 & -9 & -6 & -5 \\ -9 & -13 & -5 & -6 \\ -6 & -5 & -7 & -4 \\ -5 & -6 & -4 & -7 \end{bmatrix}$$

Let us assume that $H = 2$.

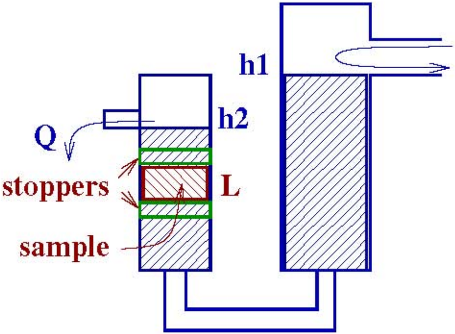
- If $Q_1 = Q_2 = 0$, then $h_i = 2$ for all h_i as expected.
- If $Q_1 = 1$ and $Q_2 = 0$, we have $h = [0.82, 1.18, 1.45, 1.55]$

- More interesting, if $Q_1 = 0$ and $Q_2 = 1$, we have

2	1.64	1.55
2	1.36	1.45

- Note that the heads depend as a linear combination of Q_1 and Q_2 and that decides the well-interference. For each unit of extraction at Q_2, there is a loss of $0.45m$ of head at cell 1.
- If K, the conductivity increases then α increases and the head-loss decreases.

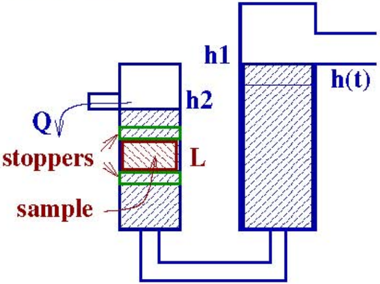
Measuring K: fixed heads



- The head difference is maintained at $h_1 - h_2$.
- The sample is held by two permeeable stoppers.
- The sample thickness is L and cross-section A.
- The system is at steady state and the outflow Q is measured.

$$K = \frac{QL}{A \cdot (h_1 - h_2)}$$

Measuring K: varying heads



- Start with height $h(0) = h_1$ and stop after time T and at height $h(T)$.
- Let cross-section of both tubes be A.
- Let Q be the total water discharged.

- We have $Q = KA(h(t) - h_2)/L$, whence we have:

$$dh/dt = -K(h(t) - h_2)/L \qquad h(0) = h_1$$

- $h(t) = (h_1 - h_2)e^{-Kt/L} + h_2$ whence we have:
- $K = L\log[(h_1 - h_2)/(h(T) - h_2)]/T$

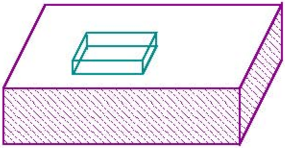
Farm-pond

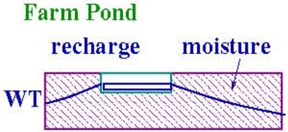
A farmer is considering a farmpond of size $10m \times 30m \times 2m$, of about Rs. 10,000 in direct and indirect costs. The objectives are:

- Recharge for better moisture for *rabi*.
- Use for paddy crop in dry spells.

Please advise.

- A real-life techno-economic problem.
- Mainly unsaturated flows (moisture) and transient analysis.
- Crop related information: wilt-points.
- Evaporation-Transpiration rates and Infiltration.
- Monsoon behaviour.





Thanks