Stability Analysis of Discrete Time Systems

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Discrete Time Models

Computer control relevant discrete models

State Space Model

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}\mathbf{u}(k)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k)$$

Transfer Function Matrix

$$\mathbf{y}(k) = G(q)\mathbf{u}(k) = \mathbf{C}\left[q\mathbf{I} - \Phi\right]^{-1}\mathbf{\Gamma}\mathbf{u}(k)$$

$$\mathbf{y}(z) = G(q)\mathbf{u}(z) = \mathbf{C}\left[z\mathbf{I} - \Phi\right]^{-1}\mathbf{\Gamma}\mathbf{u}(z)$$

Zeros

Roots of $B(z) = 0$

SISO System

Poles

Roots of $A(z) = 0$

$$y(z) = \frac{B(z)}{A(z)}u(z) = \frac{b_1 z^{n-1} + b_2 z^{n-2} + \ldots + b_n}{z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ldots + a_n}$$

Stability of Unforced System

Motivation: Controlled systems can be viewed as 'unforced systems'

Example : Consider feedback controller

$$\mathbf{u}(k) = \mathbf{G}[\mathbf{r}(k) - \mathbf{y}(k)] = \mathbf{G}[\mathbf{r}(k) - \mathbf{C}\mathbf{x}(k)]$$

$$\Downarrow$$

Closed loop equation

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \mathbf{\Gamma}\mathbf{G}[\mathbf{r}(k) - \mathbf{C}\mathbf{x}(k)]$$

$$\Downarrow$$

Closed loop dynamics for scenario : $\mathbf{r}(k) = \overline{\mathbf{0}}$

$$\mathbf{x}(k+1) = [\mathbf{\Phi} - \mathbf{\Gamma}\mathbf{G}\mathbf{C}]\mathbf{x}(k)$$

which is an unforced system

Stability of Unforced System

Motivation: Open loop systems
with permanently pre-specified inputs

Example : Consider single input system with

$$u(k) = \alpha \sin(\omega_0 kT)$$

$$\Downarrow$$

System dynamics

$$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k) + \Gamma \sin(\omega_0 kT)$$

$$\Downarrow$$

Abstract Form

$$\mathbf{x}(k+1) = \mathbf{F}[\mathbf{x}(k), k]$$

which can be viewed as 'time varying unforced system'

Stability of Autonomous Systems

Consider General Unforced System

$$\mathbf{x}(k+1) = \mathbf{F}[\mathbf{x}(k)]$$

$\mathbf{x}(0)$: Initial Condition

$\mathbf{x} \in R^n$ and $\mathbf{F}[.]$ is an $(n \times 1)$ function vector

Steady state operating point / equilibrium point

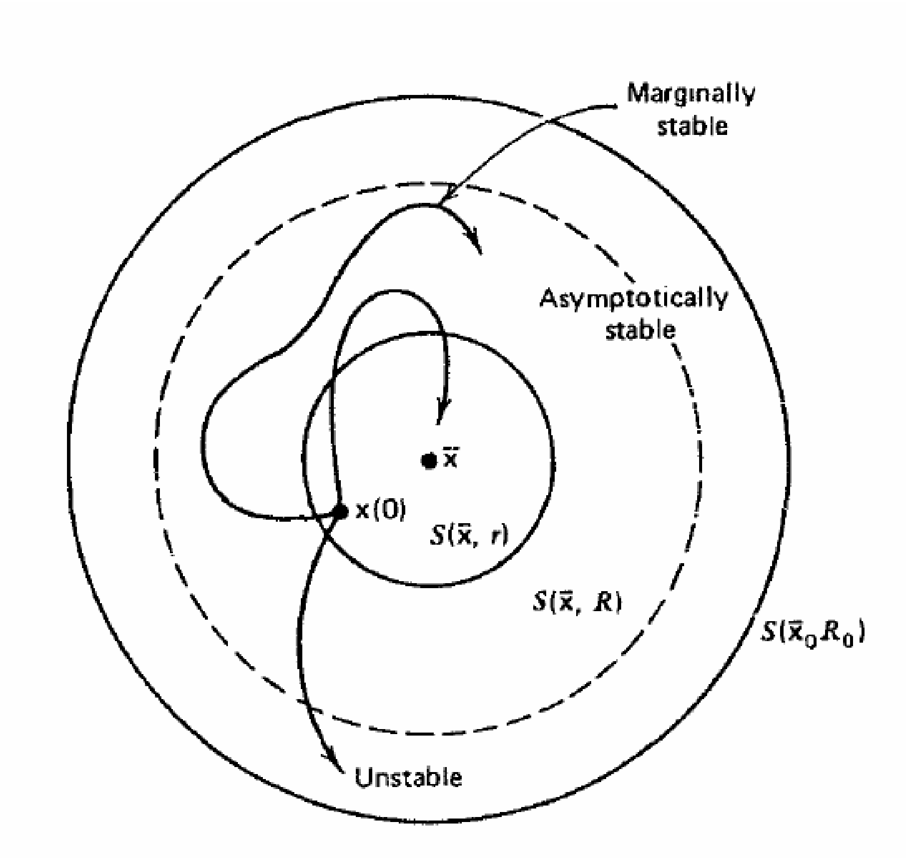
$$\overline{\mathbf{x}} = \mathbf{F}[\overline{\mathbf{x}}]$$

An equilibrium point $\overline{\mathbf{x}}$ is stable if there is an $R_0 > 0$ for which the following is true:

For every $0 < R < R_0$, there exists an $0 < r < R$, such that

if $\|\mathbf{x}(0) - \overline{\mathbf{x}}\| < r \Rightarrow \|\mathbf{x}(k) - \overline{\mathbf{x}}\| < R$ for all $k > 0$.

Stability of Autonomous Systems



An equilibrium point $\bar{\mathbf{x}}$ is asymptotically stable if it is stable and there is an $\overline{R}_0 > 0$ such that whenever $\|\mathbf{x}(k) - \bar{\mathbf{x}}\| < \overline{R}_0$ then $\mathbf{x}(k) \to \bar{\mathbf{x}}$ as $k \to \infty$

An equilibrium point $\bar{\mathbf{x}}$ is marginally stable if it is stable and not asymptotically stable.

An equilibrium point $\bar{\mathbf{x}}$ is unstable if it is NOT stable

Asymptotic Stability

Spectral radius of matrix $\mathbf{M}$, which has eigen values $\lambda_1,....\lambda_n$

$$\rho[\mathbf{M}] = \max_i |\lambda_i|$$

Unforced open loop system

$\mathbf{x}(k+1) = \mathbf{\Phi x}(k)$ with I.C. $\mathbf{x}(0)$ is asymptotically stable if $\rho\,[\mathbf{\Phi}] < 1$

Closed loop system

$\mathbf{x}(k+1) = [\mathbf{\Phi} - \mathbf{\Gamma G C}]\mathbf{x}(k)$ with I.C. $\mathbf{x}(0)$

is asymptotically stable if $\rho[\mathbf{\Phi} - \mathbf{\Gamma G C}] < 1$

Controller design problem

Given model with $(\mathbf{\Phi}, \mathbf{\Gamma}, \mathbf{C})$ matrices

choose matrix $\mathbf{G}$ such that $\rho[\mathbf{\Phi} - \mathbf{\Gamma G C}] < 1$

Marginal Stability and Instability

Unforced open loop system

$\mathbf{x}(k+1) = \mathbf{\Phi}\mathbf{x}(k)$ with I.C. $\mathbf{x}(0)$

is matginally stable if $\rho\left[\mathbf{\Phi}\right] = 1$

and unstable if $\rho\left[\mathbf{\Phi}\right] > 1$.

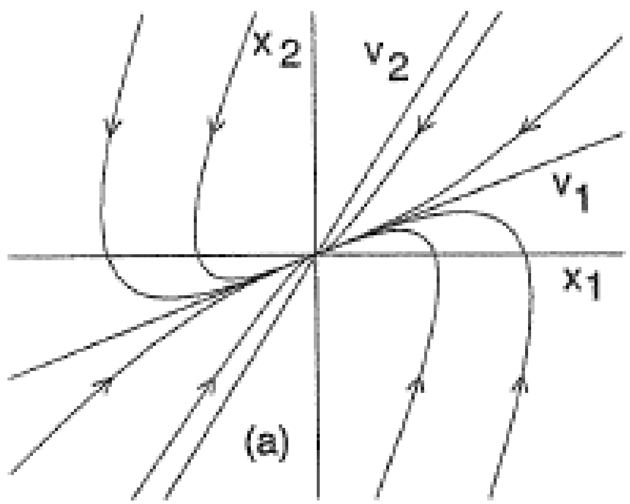
Note

When matrix Φ is obtained through linearization of a nonlinear mechanistic model ONLY local Asymptotic stability OR Instability of $\overline{\mathbf{x}}$ can be assessed using $\rho[\Phi]$.

No conclusion can be reached regarding the local dynamic behavior if $\rho[\Phi] = 1$.

Phase Portraits of 2 state System

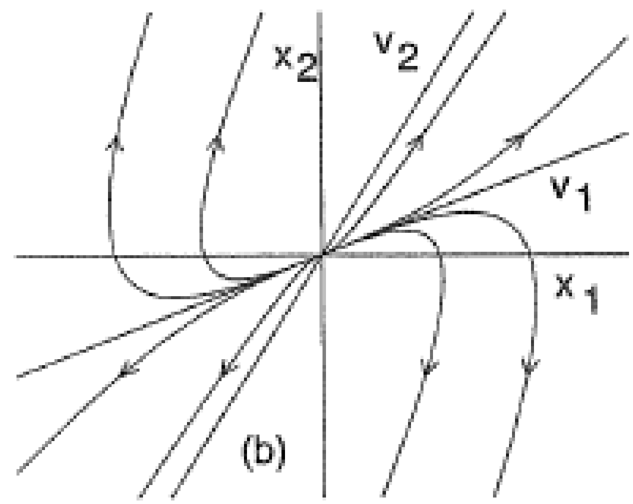
Visualization of state trajectories (starting from arbitrary initial conditions) for a **two state discrete linear autonomous** system



Stable Node

Eigen Values are real and

$\lambda_1 < 0, \lambda_2 < 0$

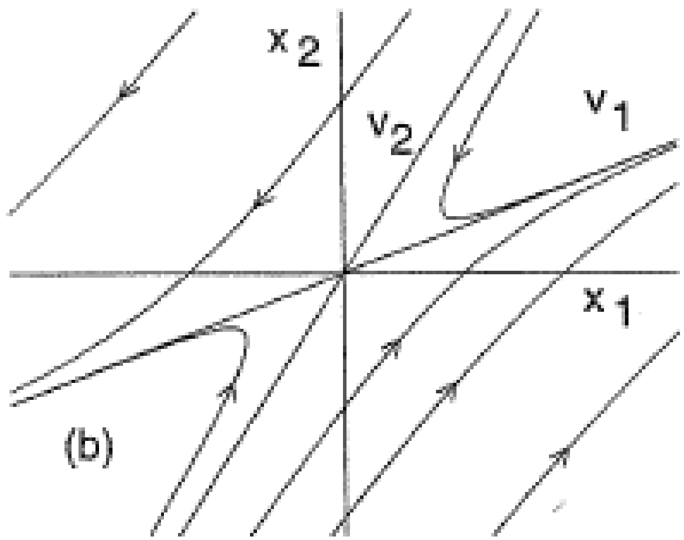


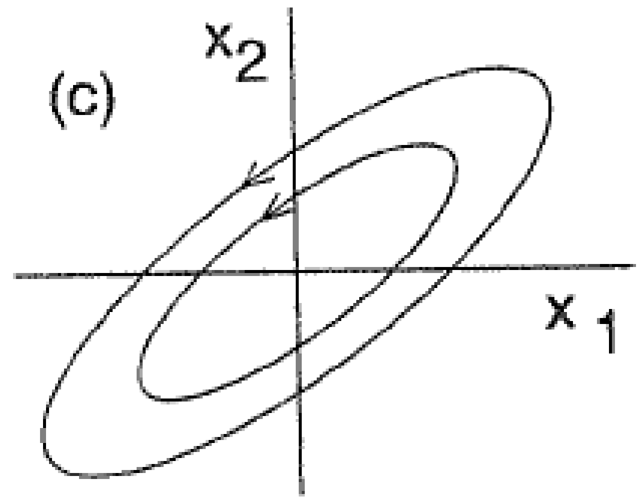
Unstable Node

Eigen Values are real and

$\lambda_1 > 0, \lambda_2 > 0$

Phase Portraits of 2 state System





Saddle Point

Eigen Values are real and

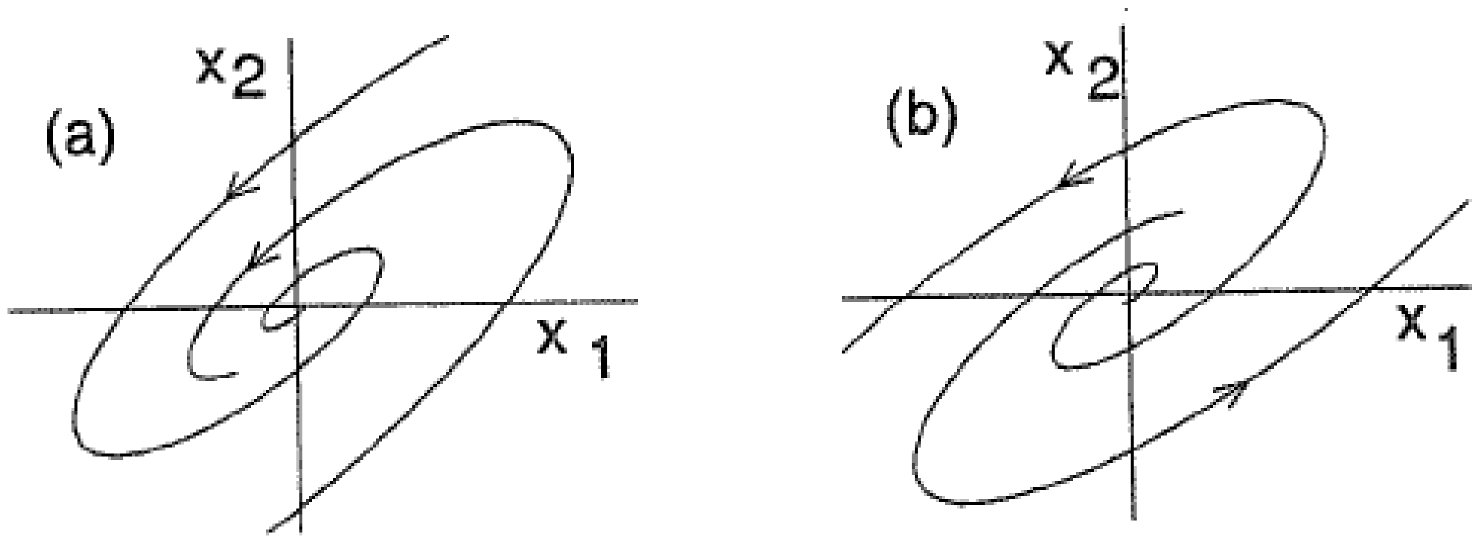
$\lambda_1 < 0, \lambda_2 > 0$

Center

Eigen Values are imaginary and

$\lambda_1 = +j\beta$ and $\lambda_2 = -j\beta$

Phase Portraits of 2 state System



Stable focus

Eigen Values are imaginary and

$\lambda_1 = \alpha + j\beta$ and $\lambda_2 = \alpha - j\beta$

with $\alpha < 0$

Unstable focus

Eigen Values are imaginary and

$\lambda_1 = \alpha + j\beta$ and $\lambda_2 = \alpha - j\beta$

with $\alpha > 0$

BIBO Stability

Bounded Input Bounded Output (**BIBO**) Stability

A linear time invariant system is BIBO stable if a bounded input produced a bounded output **<u>for every initial condition</u>**.

$\rho[\Phi] < 1 \Rightarrow$ Transfer function matrix relating y(k) with u(k) is BIBO stable

But, BIBO stability does not imply asymptotic stability

Asymptotic stability if the strongest concept. Asymptotic stability of a system implies stability and BIBO stability

Example

Consider sampled harmonic oscillator system

$$\mathbf{x}(k+1) = \begin{bmatrix} \cos(\omega T) & \sin(\omega T) \\ -\sin(\omega T) & \cos(\omega T) \end{bmatrix} \mathbf{x}(k) + \begin{bmatrix} 1-\cos(\omega T) \\ \sin(\omega T) \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} 1 & 0 \end{bmatrix} \mathbf{x}(k)$$

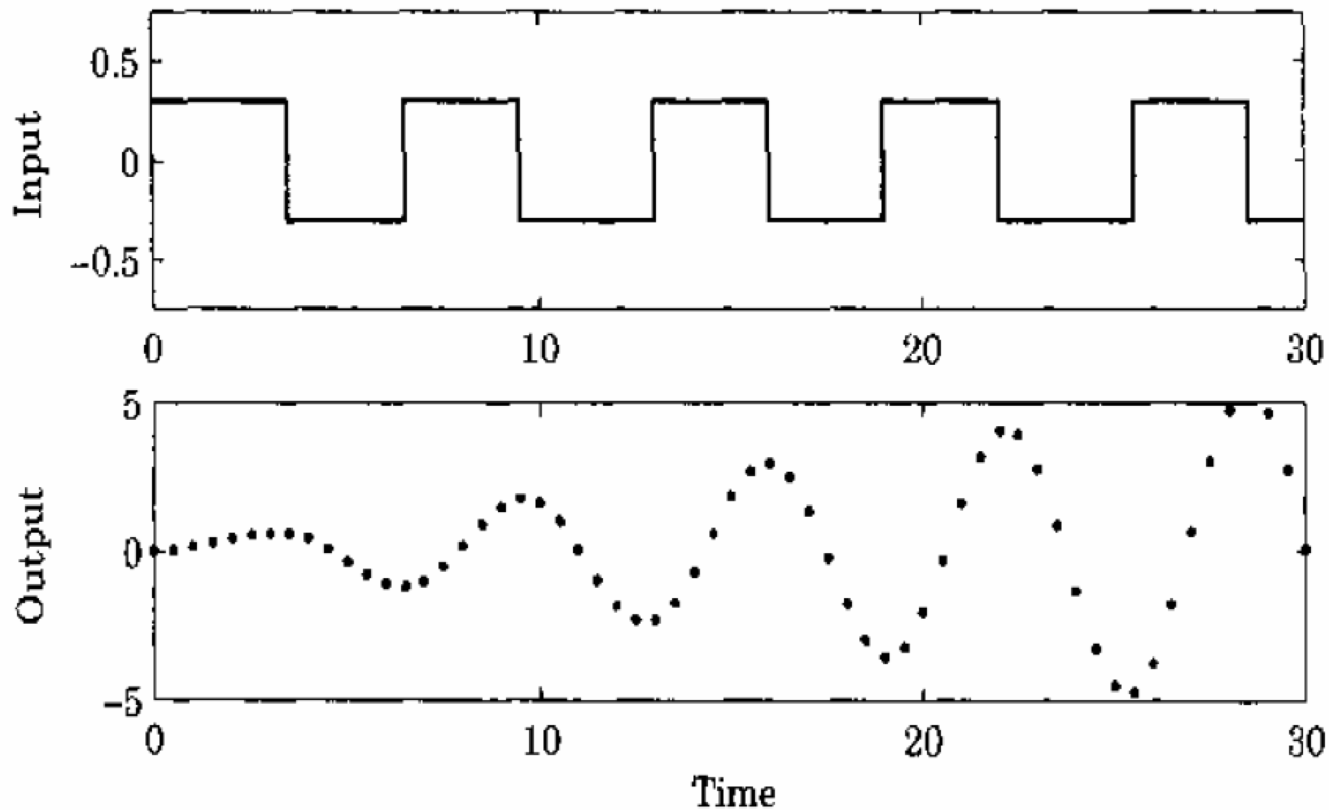
$$\rho[\Phi] = 1 \Rightarrow \text{Unforced system is stable}$$

$$\text{because } \mathbf{x}(k) = \mathbf{x}(0) \text{ if } u(k) = 0$$

Transfer function of this system has poles on the unit circle and this renders the system BIBO unstable.

Example

Input: square wave with frequency ω rad/sec



Output amplitude
grows with time and system is not BIBO stable

BIBO Stability

Consider closed loop equation

with output feedback controller

$$\mathbf{x}(k+1) = [\boldsymbol{\Phi} - \boldsymbol{\Gamma}\mathbf{G}\mathbf{C}]\mathbf{x}(k) + [\boldsymbol{\Gamma}\mathbf{G}]\mathbf{r}(k)$$

$$\mathbf{y}(k) = \mathbf{C}\mathbf{x}(k)$$

If matrix $\mathbf{G}$ is choosen such that $\rho[\boldsymbol{\Phi} - \boldsymbol{\Gamma}\mathbf{G}\mathbf{C}] < 1$

then the closed loop transfer function matrix

relating $\mathbf{y}(k)$ with $\mathbf{r}(k)$ is BIBO stable

Stability Tests

- Direct calculation of eigen values of the state transition matrix
- Methods based on properties of characteristic polynomial
- Root locus methods
- Nyquist plots
- Lyapunov's method

Jury's Stability Criterion

Consider characteristic polynomial

$$A(z) = a_0 z^n + a_1 z^{n-1} + \ldots + a_n = 0$$

$$\begin{array}{ccccc|l}
a_0 & a_1 & \ldots & a_{n-1} & a_n & \\
a_n & a_{n-1} & \ldots & a_1 & a_0 & \alpha_n = \dfrac{a_n}{a_0} \\
\hline
a_0^{n-1} & a_1^{n-1} & \ldots & a_{n-1}^{n-1} & & \\
a_{n-1}^{n-1} & a_{n-2}^{n-1} & \ldots & a_0^{n-1} & & \alpha_{n-1} = \dfrac{a_{n-1}^{n-1}}{a_0^{n-1}} \\
\hline
\vdots & & & & & \\
\vdots & & & & & \\
a_0^0 & & & & &
\end{array}$$

Construct a Table as shown here using --->

$$a_i^{k-1} = a_i^k - \alpha_k a_{k-i}^k$$

$$\alpha_k = \frac{a_k^k}{a_0^k}$$

Jury's Stability Criterion

Jury's Stability Test

If $a_0 > 0$, then the characteristic equation

has all roots inside the unit disc

if and only if all a_0^k for $k = 0,1,\ldots,n-1$ are positive.

If no a_0^k is zero, then the number of -ve a_0^k

is equal to number of roots outside the unit circle.

Necessary Conditions for Stability

If $a_0^k > 0$ for $k = 0,1,\ldots,n-1$

then the condition $a_0^0 > 0$ is equivalent to conditions

$$A(z=1) > 0$$

$$(-1)^n A(z=-1) > 0$$

Example

$$A(z) = z^2 + a_1 z + a_2 = 0$$

Jury's scheme is

$$\begin{array}{ccc|c}
1 & a_1 & a_2 & \\
a_2 & a_1 & 1 & \alpha_2 = a_2 \\
\hline
1 - a_2^2 & a_1(1 - a_2) & & \\
a_1(1 - a_2) & 1 - a_2^2 & & \alpha_1 = \dfrac{a_1}{1 + a_2} \\
\hline
1 - a_2^2 - \dfrac{a_1^2(1 - a_2)}{1 + a_2} & & &
\end{array}$$

Example

The roots are inside the unit circle if

$$1 - a_2^2 > 0$$

$$\frac{1 - a_2}{1 + a_2}\left((1 + a_2)^2 - a_1^2\right) > 0$$

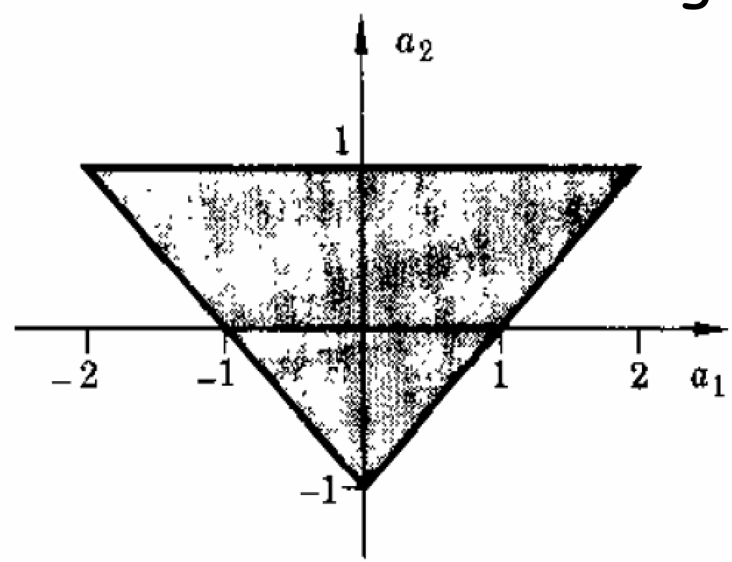
This gives the following conditions

$$a_2 < 1$$

$$a_2 > -1 + a_1$$

$$a_2 > -1 - a_1$$

Stability Region



Liapunov Function

Lyapunov function

$V(\mathbf{x}): R^n \rightarrow R$ represent a Lyapunov function for the autonomous dynamic system,

$$\mathbf{x}(k+1) = F[\mathbf{x}(k)]$$

$$F(0) = 0$$

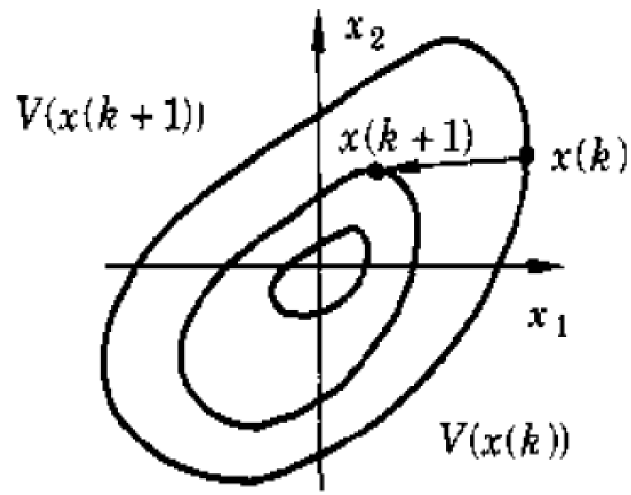
if the following conditions are satisfied

1. $V(\mathbf{x})$ is continuous in $\mathbf{x}$ and $V(0) = 0$.

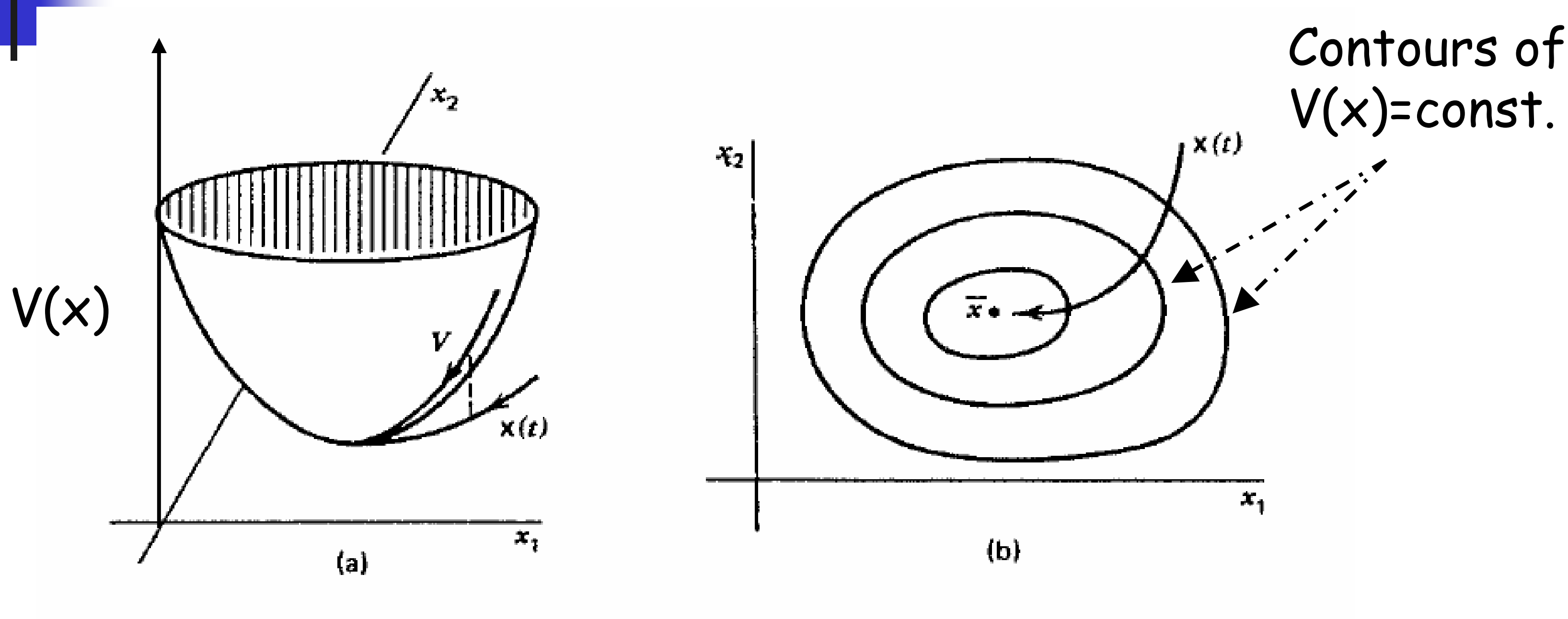
2. $V(\mathbf{x})$ is positive definite.

3. $\Delta V(\mathbf{x}) = V[\mathbf{x}(k+1)] - V[\mathbf{x}(k)]$
$= V[F(\mathbf{x}(k))] - V[x(k)]$

is negative definite for all k.



Liapunov Function



System evolves according to the its laws of dynamics.

The third condition implies that the value of Liapunov function V[x(k)] never increases with time as the system evolves in time.

Note: Liapunov function is (typically) an artificial scalar function for the state vector and in not defined by system dynamics