

MA 105: Calculus

Lecture 20

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Vector Algebra

We begin a part of this course known as '**Vector Analysis**'.

It deals with scalar fields and vector fields.

We begin with some concepts in '**Vector Algebra**'.

In Lecture 12, we have defined the **Euclidean space**

$$\mathbb{R}^m := \{\mathbf{x} = (x_1, \dots, x_m) : x_i \in \mathbb{R} \text{ for } i = 1, \dots, m\},$$

where $m \in \mathbb{N}$. An element of $\mathbb{R}^1 := \mathbb{R}$ is called a **scalar**, and an element of $\mathbb{R}^m$ is called a **vector** if $m \geq 2$.

Let $\mathbf{x} := (x_1, \dots, x_m)$, $\mathbf{y} := (y_1, \dots, y_m) \in \mathbb{R}^m$, and $a \in \mathbb{R}$. We have already defined the sum $\mathbf{x} + \mathbf{y}$ and the scalar multiple $a\mathbf{x}$. Also, we have studied the **scalar product** of $\mathbf{x}$ and $\mathbf{y}$:

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + \dots + x_m y_m \in \mathbb{R}.$$

It is often called the **dot product** of $\mathbf{x}$ and $\mathbf{y}$.

Vector Product

Let $m := 3$, $\mathbf{x} := (x_1, x_2, x_3)$ and $\mathbf{y} := (y_1, y_2, y_3)$. We now define the **vector product** of $\mathbf{x}$ with $\mathbf{y}$:

$$\mathbf{x} \times \mathbf{y} := (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1) \in \mathbb{R}^3.$$

It is often called the **cross product** of $\mathbf{x}$ with $\mathbf{y}$. Let

$$\mathbf{i} := (1, 0, 0), \quad \mathbf{j} := (0, 1, 0), \quad \mathbf{k} := (0, 0, 1).$$

Then $\mathbf{x} := x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$, $\mathbf{y} := y_1\mathbf{i} + y_2\mathbf{j} + y_3\mathbf{k}$, and

$$\mathbf{x} \times \mathbf{y} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Note: $\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$.

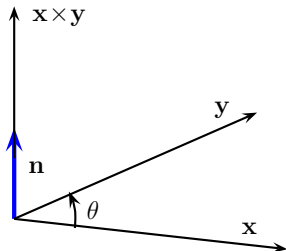
Properties of a determinant show that for every $\mathbf{z} \in \mathbb{R}^3$,

$$(\mathbf{x} + \mathbf{y}) \times \mathbf{z} = (\mathbf{x} \times \mathbf{z}) + (\mathbf{y} \times \mathbf{z}) \quad \text{and} \quad \mathbf{y} \times \mathbf{x} = -(\mathbf{x} \times \mathbf{y}).$$

Let $\mathbf{x} \neq \mathbf{0}$ and $\mathbf{y} \neq \mathbf{0}$. It can be shown that

$$\mathbf{x} \times \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| (\sin \theta) \mathbf{n},$$

where $\theta \in [0, \pi]$ is the angle between $\mathbf{x}$ and $\mathbf{y}$, and $\mathbf{n}$ is the unit vector which is perpendicular to the plane containing $\mathbf{x}$ and $\mathbf{y}$, and obeys the 'right-hand rule'.



Hence $\|\mathbf{x} \times \mathbf{y}\|$ = the area of the parallelogram with sides $\mathbf{x}$, $\mathbf{y}$.

Scalar Triple Product and Vector triple Product

Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^3$. Then $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) \in \mathbb{R}$ and $\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) \in \mathbb{R}^3$ are called the **scalar triple product** and the **vector triple product** of $\mathbf{x}, \mathbf{y}, \mathbf{z}$ respectively. It is easy to see that if $\mathbf{x} = (x_1, x_2, x_3)$, $\mathbf{y} = (y_1, y_2, y_3)$, $\mathbf{z} = (z_1, z_2, z_3)$, then

$$\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix}.$$

Geometrically, $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z})$ can be interpreted as **the (signed) volume of the parallelepiped** defined by the vectors $\mathbf{x}, \mathbf{y}, \mathbf{z}$.

One can prove the **Lagrange formula**

$$\mathbf{x} \times (\mathbf{y} \times \mathbf{z}) = (\mathbf{x} \cdot \mathbf{z}) \mathbf{y} - (\mathbf{x} \cdot \mathbf{y}) \mathbf{z}$$

by considering each component of the LHS and the RHS.

Scalar Fields and Vector Fields

A **scalar field** is an assignment of a scalar to each point in a region in the space. For example, the **temperature at a point on the earth** is a scalar field (defined on a subset of $\mathbb{R}^3$.)

A **vector field** is an assignment of a vector to each point in a region in the space. For example, the **velocity field of a moving fluid** is a vector field that associates a velocity vector to each point in the fluid.

Definition

Let $m \in \mathbb{N}$, and let D be a subset of $\mathbb{R}^m$.

A **scalar field** is a map from D to $\mathbb{R}$.

A **vector field** is a map from D to $\mathbb{R}^m$.

If $m = 2$, then it is called a **vector field in the plane**, and if $m = 3$, then it is called a **vector field in the space**.

Smooth Scalar and Vector Fields

Suppose D is an **open** subset of $\mathbb{R}^m$, that is, every point in D is an interior point of D .

A scalar field $f : D \rightarrow \mathbb{R}$ is called **smooth** if all first order partial derivatives of f exist and are continuous on D . The set of all smooth scalar fields on D is denoted by $C^1(D)$. Similarly, the set of all scalar fields on D having continuous partial derivatives of the first and second order is denoted by $C^2(D)$.

Let $\mathbf{F} : D \rightarrow \mathbb{R}^m$ be a vector field on D , and let

$$\mathbf{F}(\mathbf{x}) = (F_1(\mathbf{x}), \dots, F_m(\mathbf{x})) \quad \text{for } \mathbf{x} \in D,$$

where $F_i : D \rightarrow \mathbb{R}$ is scalar field on D , $i = 1, \dots, m$. Then the vector field $\mathbf{F}$ is called **smooth** if each component scalar field F_i is smooth, that is, if each $\frac{\partial F_i}{\partial x_j}$, $i, j = 1, \dots, m$ exists and is continuous on D .

Gradient, Divergence and Curl

Let f be a smooth scalar field on $D \subset \mathbb{R}^3$. Then the vector field

$$\text{grad } f := \nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

defined on D is called the **gradient field** of f .

Next, let $\mathbf{F} := (P, Q, R)$ be a smooth vector field on $D \subset \mathbb{R}^3$. The **divergence field** of $\mathbf{F}$ is the scalar field on D defined by

$$\text{div } \mathbf{F} := \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z},$$

and the **curl field** of $\mathbf{F}$ is the vector field on D defined by

$$\text{curl } \mathbf{F} := \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right).$$

As in the case of the cross product $\mathbf{x} \times \mathbf{y}$, we may write

$$\nabla \times \mathbf{F} = \nabla \times (P, Q, R) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}.$$

For any $m \in \mathbb{N}$, we can define the gradient field $\text{grad } f$ of a scalar field f on an open subset of $\mathbb{R}^m$ as well as the divergence field $\text{div } \mathbf{F}$ of a vector field $\mathbf{F}$ on an open subset of $\mathbb{R}^m$ in a similar manner. But the curl field $\text{curl } \mathbf{F}$ can **only** be defined for a vector field $\mathbf{F}$ on an open subset of $\mathbb{R}^3$.

Of course, if D is an open subset of $\mathbb{R}$ and f is a smooth scalar field on D , then we can let $\mathbf{F}(x, y, z) := (f(x), 0, 0)$ for $(x, y, z) \in D \times \mathbb{R}^2$, and define $\text{curl } f := \text{curl } \mathbf{F}$. Also, if D is an open subset of $\mathbb{R}^2$ and $\mathbf{G}$ is a smooth vector field on D , then we can let $\mathbf{F}(x, y, z) := (G(x, y), 0)$ for $(x, y, z) \in D \times \mathbb{R}$, and define $\text{curl } \mathbf{G} := \text{curl } \mathbf{F}$.

The GCD Sequence

Suppose the first and second order partial derivatives of f and of P, Q, R exist and are continuous on D . By the Mixed Partials Theorem, we obtain (i) $\text{curl}(\text{grad } f) = \nabla \times (\nabla f) = \mathbf{0}$:

$$\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}, \frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}, \frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x} \right) = (0, 0, 0),$$

and also (ii) $\text{div}(\text{curl } \mathbf{F}) = \nabla \cdot (\nabla \times \mathbf{F}) = 0$:

$$\frac{\partial}{\partial x} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) + \frac{\partial}{\partial y} \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) + \frac{\partial}{\partial z} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = 0.$$

Thus **G**radient, **C**url and **D**ivergence give a sequence of maps

$$\left\{ \begin{array}{c} \text{scalar} \\ \text{fields} \end{array} \right\} \xrightarrow{\text{grad}} \left\{ \begin{array}{c} \text{vector} \\ \text{fields} \end{array} \right\} \xrightarrow{\text{curl}} \left\{ \begin{array}{c} \text{vector} \\ \text{fields} \end{array} \right\} \xrightarrow{\text{div}} \left\{ \begin{array}{c} \text{scalar} \\ \text{fields} \end{array} \right\}$$

which satisfies $\text{curl}(\text{grad } f) = \mathbf{0}$ and $\text{div}(\text{curl } \mathbf{F}) = 0$, so that the successive composites are zero.

The above phenomenon raises the following basic questions:

(i) If $\mathbf{G}$ is a smooth vector field such that $\text{curl } \mathbf{G} = \mathbf{0}$, then is $\mathbf{G}$ a gradient field, that is, is there a scalar field f such that $\mathbf{G} = \text{grad } f$?

(ii) If $\mathbf{H}$ is smooth vector field such that $\text{div } \mathbf{H} = 0$, then is $\mathbf{H}$ a curl field, that is, is there a vector field $\mathbf{F}$ such that $\mathbf{H} = \text{curl } \mathbf{F}$?

These questions are reminiscent of the Fundamental theorem of Calculus, Part I, which answers the following question in the affirmative: If g is a continuous function on $[a, b]$, then is there an antiderivative of g , that is, is there a differentiable function f such that $g = f'$?

As such, the two questions raised above call for a suitable theory of integration, to which we now turn. Eventually, we shall come back to these questions.

Laplacian

Let f be a smooth vector field on $D \subset \mathbb{R}^3$, and suppose the second order partials f_{xx}, f_{yy}, f_{zz} exist on D . Let us consider the maps

$$\left\{ \begin{array}{c} \text{scalar} \\ \text{fields} \end{array} \right\} \xrightarrow{\text{grad}} \left\{ \begin{array}{c} \text{vector} \\ \text{fields} \end{array} \right\} \xrightarrow{\text{div}} \left\{ \begin{array}{c} \text{scalar} \\ \text{fields} \end{array} \right\}.$$

The **Laplacian field** of f is the scalar field defined on D by

$$\text{div}(\text{grad } f) := \nabla \cdot (\nabla f) = \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

The Laplacian plays a very important role in the theory of partial differential equations, and its various applications.

Paths

Let $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, and let $m \in \mathbb{N}$. A **path** or a **parametrized curve** in $\mathbb{R}^m$ is a continuous map $\gamma : [\alpha, \beta] \rightarrow \mathbb{R}^m$, that is, if $\gamma = (\gamma_1, \dots, \gamma_m)$, then $\gamma_j : [\alpha, \beta] \rightarrow \mathbb{R}$ is continuous for each $j = 1, \dots, m$.

A path $\gamma : [\alpha, \beta] \rightarrow \mathbb{R}^m$ is called **closed** if $\gamma(\alpha) = \gamma(\beta)$. Such a path is also called a **loop**. A path $\gamma : [\alpha, \beta] \rightarrow \mathbb{R}^m$ is called **simple** if $\gamma(t_1) \neq \gamma(t_2)$ for all $t_1, t_2 \in (\alpha, \beta)$, $t_1 \neq t_2$.

If for $t \in [\alpha, \beta]$,

$$\frac{d\gamma}{dt} = \gamma'(t) := (\gamma'_1(t), \dots, \gamma'_m(t))$$

exists, then it is called the **tangent vector** to γ at t , and if it is nonzero, then $\hat{\mathbf{t}} := \gamma'(t)/\|\gamma'(t)\|$ is called the **unit tangent vector** to γ at t . (We write $\hat{\mathbf{t}}$ instead of $\hat{\mathbf{t}}_\gamma(t)$ for brevity.)

Further, a path γ in $\mathbb{R}^m$ is called **smooth** or C^1 if each $\gamma_j : [\alpha, \beta] \rightarrow \mathbb{R}$ is continuously differentiable for $j = 1, \dots, m$; in case γ is a closed curve, that is, $\gamma(\alpha) = \gamma(\beta)$, we also require $\gamma'(\alpha) = \gamma'(\beta)$.

A smooth path γ in $\mathbb{R}^m$ is called **regular** if $\gamma'(t) \neq \mathbf{0}$ for all $t \in [\alpha, \beta]$, that is, if the unit tangent vector to γ exists at each $t \in [\alpha, \beta]$.

A path γ in $\mathbb{R}^m$ is called **piecewise smooth** if there are $\alpha := t_0 < t_1 < \dots < t_n =: \beta$ such that γ is smooth on each $[t_{i-1}, t_i]$, $i = 1, \dots, n$, and it is called **piecewise regular**, if γ is regular on each $[t_{i-1}, t_i]$, $i = 1, \dots, n$.

We shall assume hereafter that all paths are piecewise smooth, unless otherwise stated.

Path-connected and Convex Subsets

A subset D of $\mathbb{R}^m$ is called **path-connected** if for every $\mathbf{u}, \mathbf{v} \in D$, there is a path $\gamma : [\alpha, \beta] \rightarrow \mathbb{R}^m$ such that $\gamma(\alpha) = \mathbf{u}$, $\gamma(\beta) = \mathbf{v}$ and $\gamma(t) \in D$ for all $t \in (\alpha, \beta)$.

In particular, a **convex** subset D of $\mathbb{R}^m$ is path-connected since for $\mathbf{u}, \mathbf{v} \in D$, there is the straight-line path $\gamma : [0, 1] \rightarrow \mathbb{R}^m$ defined by $\gamma(t) := \mathbf{u} + t(\mathbf{v} - \mathbf{u}) \in D$ for $t \in [0, 1]$.

Examples:

- (i) The subset $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ of $\mathbb{R}^2$ is path-connected; in fact it is convex.
- (ii) The subset $\{(x, y) \in \mathbb{R}^2 : 1/2 \leq x^2 + y^2 \leq 1\}$ of $\mathbb{R}^2$ is path-connected, but it is not convex.
- (iii) The subset $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \cup \{(2, 0)\}$ of $\mathbb{R}^2$ is not path-connected.

Example

Let $a > 0$. Define $\gamma(t) := (a \cos t, a \sin t)$ for $t \in [-\pi, \pi]$.

This path is called the **standard parametrized circle in $\mathbb{R}^2$ of radius a** . Since $\gamma(-\pi) = \gamma(\pi)$ and

$\gamma'(t) := (-a \sin t, a \cos t)$ for $t \in [-\pi, \pi]$, we see that γ is closed, simple, smooth, regular, and its unit tangent vector at $t \in [-\pi, \pi]$ is $\hat{\mathbf{t}} := (-\sin t, \cos t)$.

If we let $\tilde{\gamma}(t) := (a \cos 2t, a \sin 2t)$ for $t \in [-\pi, \pi]$, then it is easy to see that $\tilde{\gamma}$ is also closed, smooth, regular, and its unit tangent vector at $t \in [-\pi, \pi]$ is $(-\sin 2t, \cos 2t)$.

Note that $\tilde{\gamma}([-\pi, \pi]) = \gamma([-\pi, \pi])$, that is, the functions $\tilde{\gamma}$ and γ have the same range, namely $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$, although they are clearly different paths: one goes around the circle once, but the other does so twice.